Graphon estimation beyond binary edges

POPNets Workshop, 2024

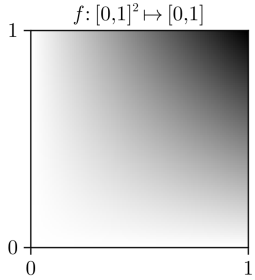
Charles Dufour
Statistical Data Science group, Prof. Sofia Olhede
Institue of Mathematics, Ecole Polytechnique Fédérale de Lausanne

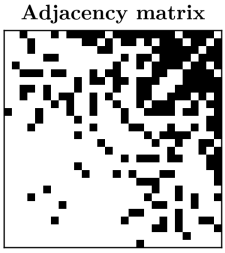
Tuesday 18th June, 2024

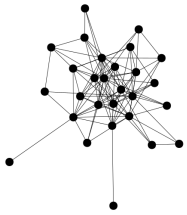
Overview

1. Simple graphs and graphons
2. Decorated graphs and decorated graphons
3. Graphon estimation

Graphon for simple exchangeable graphs



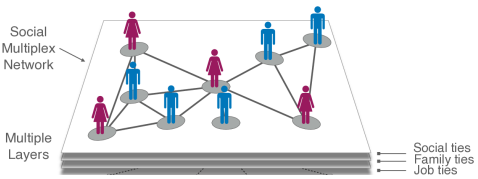


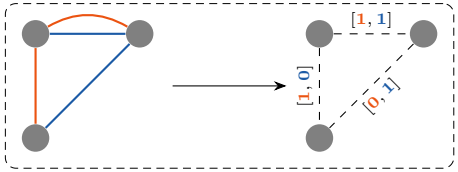


$$A_{ij}|\xi_i, \xi_j \sim \text{Bernoulli}\left(f(\xi_i, \xi_j)\right)$$

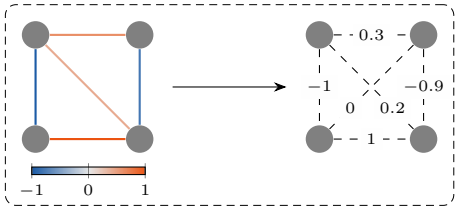
Hoover (1979); Aldous (1981)

Edges can carry more than just binary information





$\mathcal{K} = \{[0, 0], [1, 0], [0, 1], [1, 1]\}$

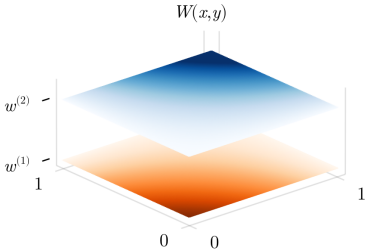


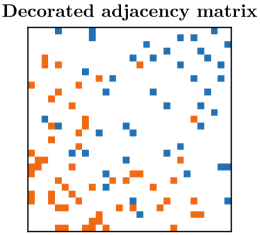
$\mathcal{K} = [-1, 1]$

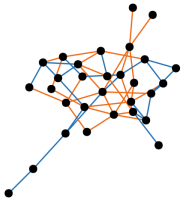
$A_{ij} \in \mathcal{K}$, space of *decorations*.

Image from De Domenico (2020)

Beyond binary edges: decorated graphons





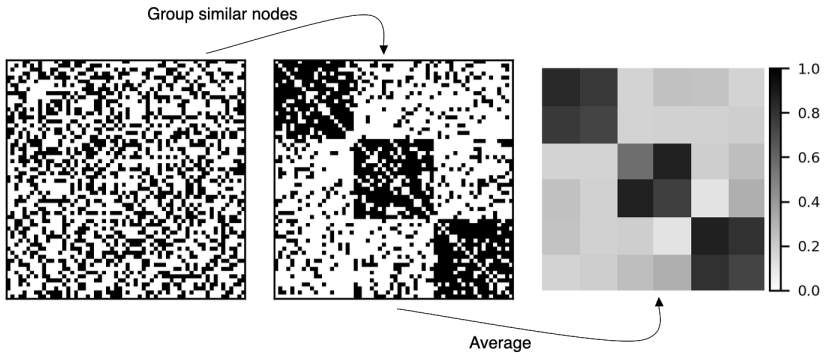


$$A_{ij}|\xi_i,\xi_j \overset{iid}{\sim} W(\xi_i,\xi_j)$$

$$\mathbb{P}\left[A_{ij} \mid \xi_i,\xi_j\right] = w^{(1)}(\xi_i,\xi_j) \quad \text{and} \quad \mathbb{P}\left[A_{ij} \mid \xi_i,\xi_j\right] = w^{(2)}(\xi_i,\xi_j)$$

Lovász and Szegedy (2010)

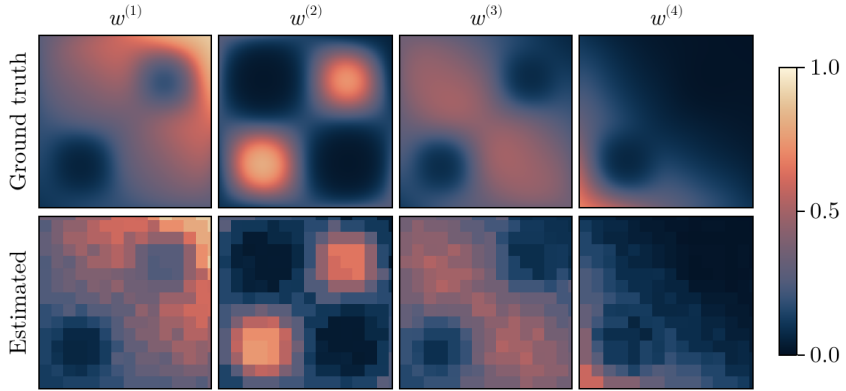
Graphon estimation for simple exchangeable graphs



Clustering of nodes based on similar behaviour

Figure: Idea of Network Histogram approximation method from Olhede and Wolfe (2014)

We can do the same for decorated graphs



(Dufour and Olhede, 2024): $n = 400$, Decorated Stochastic shape model with $(s, k) = (37, 18)$

Summary

1. Consistent estimation of generating mechanism for decorated graphs
2. Theoretical guarantees on rates of convergence confirmed by simulations
3. Framework includes* multiplex, weighted, signed, temporal, ...

* conditions may apply

References

Aldous, D. J. (1981). Representations for partially exchangeable arrays of random variables. *Journal of Multivariate Analysis*, 11(4):581–598. Publisher: Elsevier.

De Domenico, M. (2020). Multilayer networks illustrated.

Dufour, C. and Olhede, S. (2024+). Inference for decorated graphs and application to multiplex networks.

Hoover, D. N. (1979). Relations on Probability Spaces and Arrays of Random Variables. *Preprint, Institute for Advanced Study, Princeton*.

Lovász, L. and Szegedy, B. (2010). Limits of compact decorated graphs. arXiv:1010.5155 [math].

Olhede, S. C. and Wolfe, P. J. (2014). Network histograms and universality of blockmodel approximation. *Proceedings of the National Academy of Sciences*, 111(41):14722–14727. Publisher: Proceedings of the National Academy of Sciences.