A unifying framework for graph models beyond binary edges

***Inference for decorated graphs and application to multiplex networks* [1]**

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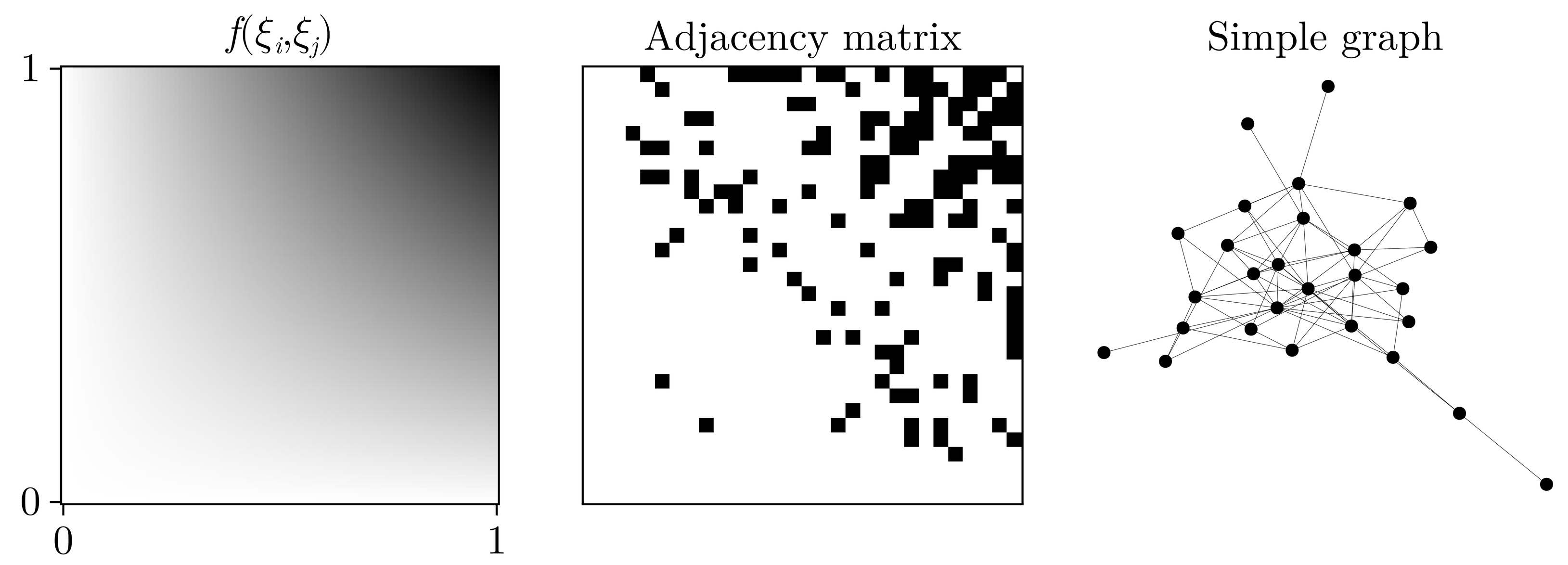
Motivation

- Modelling Complex Systems: **Graphons** provide a way to **model and analyse** complex systems where data points are interconnected, such as social networks, biological systems, and communication networks.
- Beyond Simple Data: Traditional graphons **only** handle **binary interactions**, **missing** the more **complex relationships** found in empirical data.
- Unmet Need: Decorated graphons address this complexity, but **no estimation methods** existed — until now.
- Our contribution: We present a **novel** decorated graphon **estimation technique** that allows for **complex relationship modelling** and enhances system understanding.
- Practical Use: These advancements enable better analysis of diverse data types, from social networks to protein interactions and disease relationships.

Graphs with binary edges

Simple graph: adjacency matrix $[A_{ij}] \in \{0,1\}^{n\times n}$.

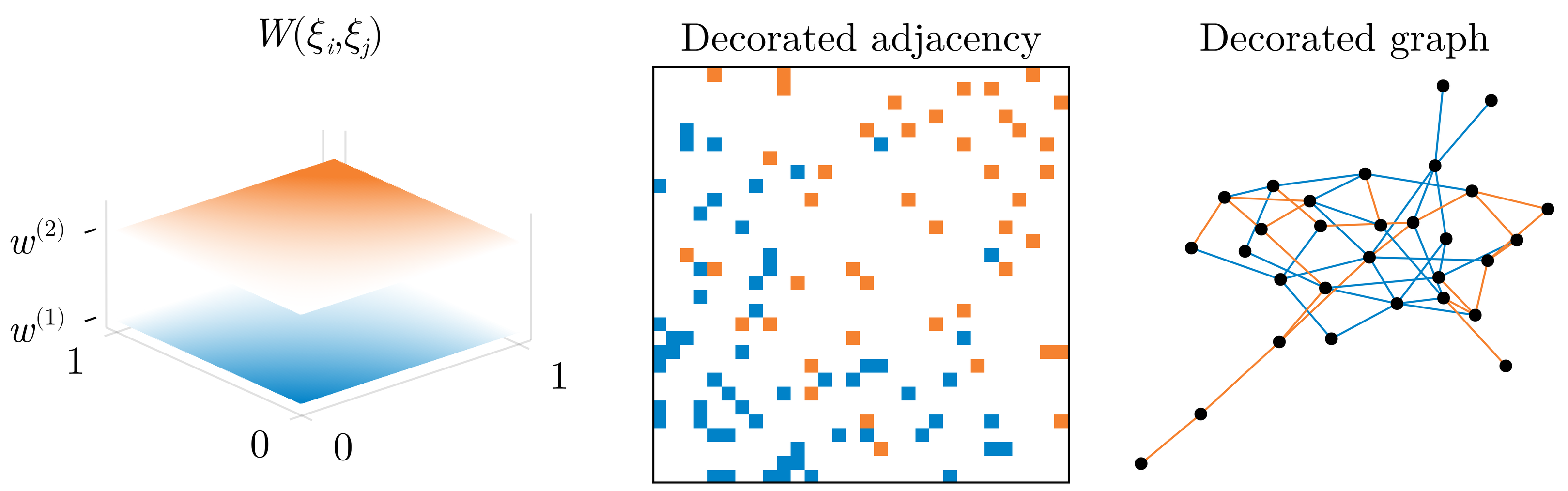


Let $\xi_i \overset{\text{iid}}{\sim} U(0,1)$, we model $[A_{ij}]$ with a *graphon* f

$$A_{ij}|\xi_i,\xi_j \overset{\text{ind.}}{\sim} \text{Bernoulli}(\theta_{ij}), \quad \theta_{ij} = f(\xi_i,\xi_j).$$

Decorated graphs (edge attributed)

For a finite set $\mathcal{K}$ with $L < \infty$ elements (e.g. $\{0,1,2\}$), we let $[A_{ij}] \in \mathcal{K}^{n\times n}$.

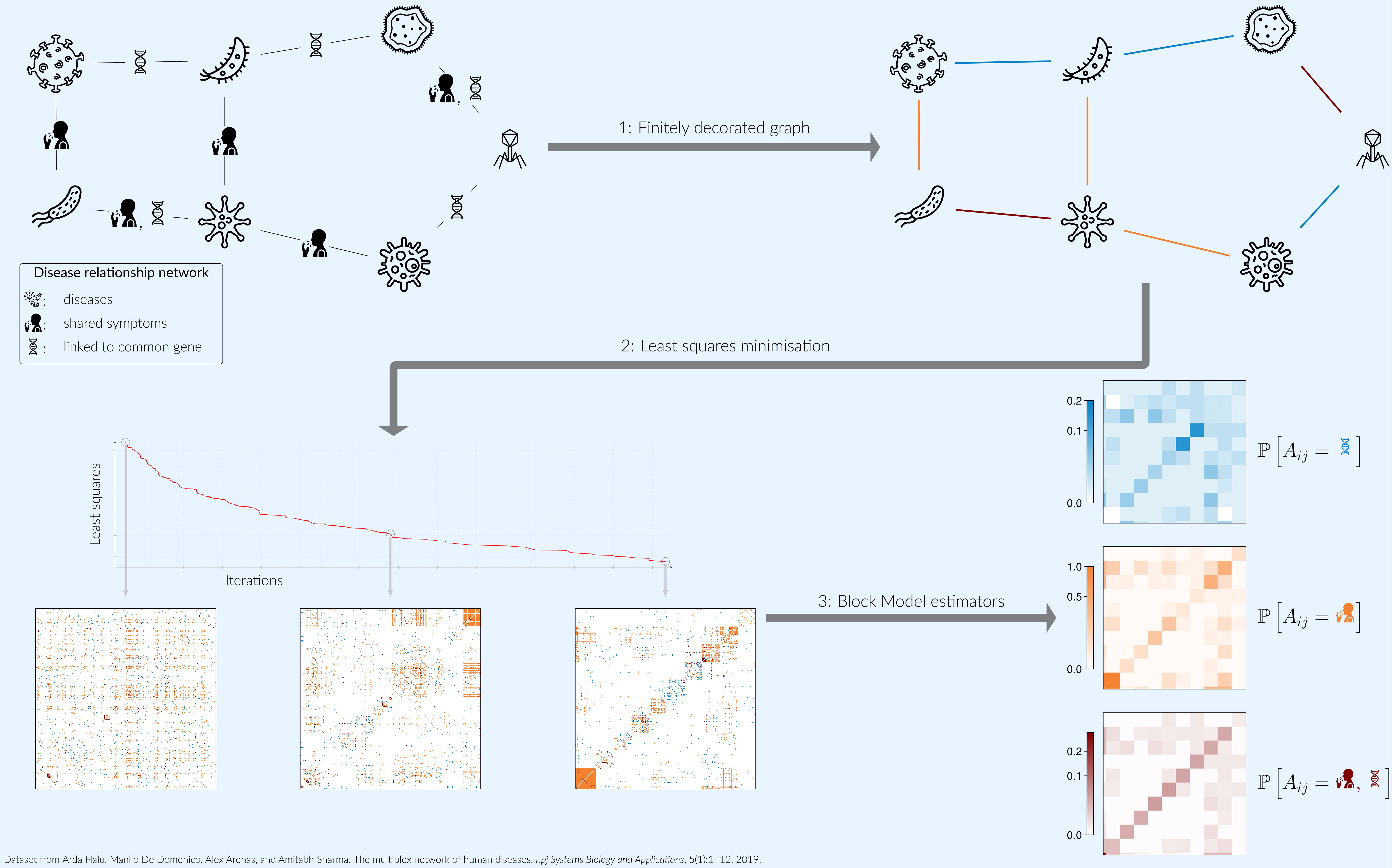


We model $[A_{ij}]$ with a *decorated graphon* $W = (w^{(1)},\dots,w^{(L)})$

$$\mathbb{P}[A_{ij} \mid \xi_i,\xi_j] = w^{(1)}(\xi_i,\xi_j) \quad \text{and} \quad \mathbb{P}[A_{ij} \mid \xi_i,\xi_j] = w^{(2)}(\xi_i,\xi_j),$$

$$\theta_{ij} = \left[w^{(l)}(\xi_i,\xi_j)\right] \in [0,1]^L.$$

Proposed methodology



Dataset from Arda Halu, Manlio De Domenico, Alex Arenas, and Amitabh Sharma. The multiplex network of human diseases. *npj Systems Biology and Applications*, 5(1):1–12, 2019.

Convergence results

- W is a k-block Block Model: [1, Theorem 1]

$$\|\theta^* - \hat\theta\|_F^2 = O_p\left(\frac{Lk^2}{n^2} + \frac{\log(k)}{n}\right).$$

- W is α-Hölder continuous: [1, Theorem 2]

$$\|\theta^* - \hat\theta\|_F^2 = O_p\left(n^{-2\alpha/(\alpha+1)} + \frac{\log(n)}{n}\right),$$

$$\text{MISE}\left(\widehat{W}, W\right) = O_p\left(n^{-2\alpha/(\alpha+1)} + \frac{\log(n)}{n} + n^{-\alpha\wedge 1}\right).$$

Takeaways

- **First estimation method** for the generating mechanism of exchangeable decorated graphs.
- The framework is **applicable to**
  1. Multiplex networks
  2. Weighted networks
  3. Signed networks
  4. Temporal networks.
- **Community detection** with **multiple types** of connections.

[1] Charles Dufour and Sofia C. Olhede. Inference for decorated graphs and application to multiplex networks, August 2024. arXiv:2408.12339 [cs, stat]

[2] Dávid Kunszenti-Kovács, László Lovász, and Balázs Szegedy. Multigraph limits, unbounded kernels, and Banach space decorated graphs. *Journal of Functional Analysis*, 282(2):109284, January 2022

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